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A CONTINUUM MODEL FOR FINITE VOID GROWTH AROUND SPHERICAL INCLUSION

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Abstract—Within the bounds of large deformation theory the necessary mathematical analysis for incompressible and nearly incompressible elastic materials is briefly stated and some results of structural stress and deformation analysis, simulating void formation in filled elastomers, are given. Void development around spherical inclusion in stretched elastomeric matrix under external pressure, and without it, is analysed.

NOMENCLATURE

Kinematic and geometric quantities

u	displacement vector with covariant u_p and contravariant u^p components
u^n_p, u^{np}	value of u_p and u^p in n-node
ξ^{np}	increment of u^{np}
ψ_n, φ_n	shape functions for n-node
$\psi_{n,i}$	derivative of the shape function ψ_n on q^i generalized coordinate
g	metric tensor of undeformed volume with components g_{ij}, g^{ij} and g^j_i
g	determinant of covariant components of the tensor **g**
F, $\mathbf{F}^T$	deformation gradient tensor and its transposition
$\mathbf{G}^x$, $\mathbf{G}^{x-1}$	Cauchy–Green deformation measure tensor and its inverse quantity with components G_{ij} and G^{ij}, respectively; $\mathbf{G}^x = \mathbf{F}^t\mathbf{F}$
G	determinant of covariant components of the tensor $\mathbf{G}^x$
$\overset{i}{I}_j$, I_3	invariants of the tensor $\mathbf{G}^x$ ($i = 1, 2, 3$; $j = 1, 2$)
j_i	invariants of Cauchy–Green deformation tensor ($i = 1, 2, 3$)
r, **R**	initial and actual radius-vectors of a material point
$\mathbf{r}_i$, $\mathbf{R}_i$	initial and actual frames of reference; $\mathbf{r}_i = \partial\mathbf{r}/\partial q^i$, $\mathbf{R}_i = \partial\mathbf{R}/\partial q^i$
$\mathbf{e}_i$	orthonormal frame of reference
Γ^k_{ij}	2nd order Kristoffel symbol
s	shear parameter
λ	elongation factor
W_0	vertical displacement of the cylinder ends
L	cylinder length
ε^c_z	measure of cylinder external deformation
V_0, S^0_p	unstrained volume and surface of the whole body
$\overset{(e)}{V}_0$, $\overset{(e)}{S}{}^0_p$	unstrained volume and surface of one finite element
n_i	covariant components of unit vector **n** normal to undeformed surface S^0_p.

Force quantities

T	true stress tensor with t^{ij} components for basis $\mathbf{e}_i$: $\mathbf{T} = t^{ij}\mathbf{e}_i\mathbf{e}_j$
Q	energy stress tensor with Q^{ij} components
σ^{ij}	contravariant components of generalized 2nd (symmetric) Piola–Kirchhoff stress tensor
$\sigma^{ij}_{,\xi^{nk}}$	derivative of quantity σ^{ij} on ξ^{nk}
K, **P**	volume and surface forces corresponding to V_0-volume and S^0_p-surface
K^p, P^p	contravariant components of **K** and **P**
F^p	contravariant components of surface forces acting on a current surface
B^p, S^p	increments of K^p and F^p
p	pressure acting on a cylinder surface
ρ^0	density of material in V_0-volume
H	normalized quantity of σ
H^n	value of H in n-node
η^n	increment of H^n.

Energy quantities

W_i, $\overset{i}{W}$	elastic pontentials ($i = 1, 2, 3$)
He	functional of Herrmann type for incompressible and nearly incompressible materials with finite deformations
A^p_{ijn}, B^p_{ijkmn}	coefficients defining the elementary work of the internal forces $\delta A = \sigma^{ij}\delta G_{ij}$ in V_0-volume: $\delta A = \sigma^{ij}\delta G_{ij} = 2\sigma^{ij}(A^p_{ijn} + B^p_{ijkmn}u^{mk})\delta u^n_p$

δ symbol of variation
A normalization factor.

Elastic moduli

$\overset{i}{C}_j, \overset{i}{\sigma}$ generalized elastic moduli ($i = 1, 2, 3$; $j = 1, 2$)
$\overset{i}{k}_j, \overset{i}{p}_j, \overset{i}{q}_j$ coefficients for $\overset{i}{C}_j$ expansion in series
$\overset{i}{\chi}_1, \overset{i}{\chi}_2$ coefficients for $\overset{i}{\sigma}$ expansion in series, $\overset{i}{\alpha} = 1/\overset{i}{\chi}_2$
χ^* normalized quantity of χ_1
Λ, B Lamé constant and volumetrical modulus
G, ν shear modulus and Poisson ratio
^{0} symbol over a letter defining quantity with zero increment.

1. INTRODUCTION

The behavior of many composite materials such as filled rubbers is very complicated for a variety of reasons. The damage accumulation due to matrix detachment from filler particles seems to be the most significant among them.

The peculiarities of the stress–strain state in the matrix detached from spherical inclusion have not yet been discussed. This task, however, seems to be important for those who investigate structure-property problems in the field of granular elastomeric composites.

Clear understanding of the stress–strain state of damaged elements can help the development of the rational constitutive relations for composite materials. The purpose of this paper is to provide new insight into the problem mentioned.

The solution is carried out within the framework of the finite elasticity theory for incompressible and nearly incompressible rubbery materials. Unified constitutive relations for slightly compressible and incompressible materials are developed and a functional for numerical solutions is suggested.

2. CONSTITUTIVE EQUATION FOR INCOMPRESSIBLE AND NEARLY INCOMPRESSIBLE ELASTOMERS

The ratio of shear modulus to volume compression modulus is extremely small in the elastomeric materials. This allows for elastomers to be considered as incompressible or nearly incompressible materials. The materials incompressibility is generally represented by the kinematic relation $I_3 = 1$, where I_3 is the third tensor invariant of the Cauchy–Green deformation measure $\mathbf{G}^x = \mathbf{F}^T\mathbf{F}$, where $\mathbf{F}$ is the deformation gradient. Our approach is based on using the generalized elasticity moduli which are thought to be the best indicators of material compressibility or incompressibility. Our approach also allows for the elaboration of a unified representation both for incompressible and slightly compressible materials. It is essential for the latter case as approximation of slightly compressible materials by incompressible ones is not valid under constrained deformation conditions.

For an isotropic homogeneous material, the elastic potential W is a function of three invariants of the Cauchy–Green deformation measure $\mathbf{G}^x$ and usually for a slightly compressible material W is expanded into a series of I_3 in the vicinity of unity with confinement up to the 2nd order (Ogden, 1984, 1986). The choice for W of one or other invariants as independent variables leads to different constitutive equations, the generalized elasticity moduli of which are defined by the elastic potential or on the basis of experimental data. In the first case the use in W of any invariant is of no importance but in the second case this may considerably simplify a problem of generalized elasticity moduli identification. Consider the three groups of tensor $\mathbf{G}^x$ invariants:

$$\overset{1}{I}_1(\mathbf{G}^x) = \mathbf{g}\cdot\cdot\mathbf{G}^x, \quad \overset{1}{I}_2(\mathbf{G}^x) = \tfrac{1}{2}\left[\overset{1}{I}{}_1^2 - \overset{1}{I}_1(\mathbf{G}^{x2})\right],$$

$$I_3(\mathbf{G}^x) = \tfrac{1}{3}\left[\overset{1}{I}_1(\mathbf{G}^{x3}) - \overset{1}{I}_1\overset{1}{I}_1(\mathbf{G}^{x2}) + \overset{1}{I}_1\overset{1}{I}_2\right]; \tag{1a}$$

$$\overset{2}{I}_1 = \overset{1}{I}_1/I_3^{1/3}, \quad \overset{2}{I}_2 = \overset{1}{I}_2/I_3^{2/3}, \quad \overset{2}{I}_3 = I_3; \tag{1b}$$

$$\overset{3}{I}_1 = \overset{1}{I}_1-(I_3-1), \quad \overset{3}{I}_2 = \overset{1}{I}_2-2(I_3-1), \quad \overset{3}{I}_3 = I_3. \tag{1c}$$

The first group comprises the well-known main invariants of the Cauchy–Green deformation measure tensor (Lurje, 1970). The invariants $\overset{2}{I}_1$ and $\overset{2}{I}_2$ are introduced by Penn (1970) and related to the tensor $\mathbf{G}^{\times}/I_3^{1/3}$ (Palmov, 1976), with only material shape changing as the third invariant of it is equal to unity. Invariants (1c) are introduced by Cescotto and Fonder (1975). They follow from the well-known expressions (Lurje, 1970) relating the main invariants of the Cauchy–Green deformation measure tensor to the main invariants of the Cauchy–Green deformation tensor j_k $(k = 1,2,3)$:

$$\overset{1}{I}_1 = 2j_1+3, \quad \overset{1}{I}_2 = 4j_1+4j_2+3, \quad I_3 = 2j_1+4j_2+8j_3+1. \tag{2}$$

Defining, from the last equality, the quantity $2j_1$ and substituting it into the first two relations we have

$$\overset{1}{I}_1 = 3-4j_2-8j_3+(I_3-1), \quad \overset{1}{I}_2 = 3-4j_2-16j_3+2(I_3-1).$$

Introducing the following notation:

$$\overset{3}{I}_1 = 3-4j_2-8j_3, \quad \overset{3}{I}_2 = 3-4j_2-16j_3,$$

we come to the invariants (1c). For small deformations, invariants (1b) and (1c) are similar. Indeed

$$\overset{2}{I}_1 = \frac{\overset{1}{I}_1}{I_3^{1/3}} = \overset{1}{I}_1[1-\tfrac{1}{3}(I_3-1)+\cdots] = \overset{1}{I}_1-(I_3-1)-\tfrac{1}{3}\left(\overset{1}{I}_1-3\right)(I_3-1)+\cdots,$$

$$\overset{2}{I}_2 = \frac{\overset{1}{I}_2}{I_3^{2/3}} = \overset{1}{I}_2[1-\tfrac{2}{3}(I_3-1)+\cdots] = \overset{1}{I}_2-2(I_3-1)-\tfrac{2}{3}\left(\overset{1}{I}_2-3\right)(I_3-1)+\cdots.$$

Here the expansions of quantities $1/I_3^{1/3}$ and $1/I_3^{2/3}$ into the series of I_3 in the vicinity of unity are used. Since the quantities $\overset{1}{I}_1-3$, $\overset{1}{I}_2-3$ and I_3-1 are of the same order, then keeping only the first order terms in the above written expressions we find that $\overset{2}{I}_1 = \overset{3}{I}_1$ and $\overset{2}{I}_2 = \overset{3}{I}_2$.

Using the invariants (1a)–(1c) elastic potential is written as

$$W_1\left(\overset{1}{I}_1,\overset{1}{I}_2,I_3\right) = W_1\left(\overset{2}{I}_1 I_3^{1/3},\overset{2}{I}_2 I_3^{2/3},I_3\right) = W_1\left(\overset{3}{I}_1+(I_3-1),\overset{3}{I}_2+2(I_3-1),I_3\right)$$
$$= W_2\left(\overset{2}{I}_1,\overset{2}{I}_2,I_3\right) = W_3\left(\overset{3}{I}_1,\overset{3}{I}_2,I_3\right)$$

and its expansion into the series of I_3 leads to the expressions:

$$W_i = \overset{i}{W}+\overset{i}{\chi}_1(I_3-1)+\tfrac{1}{2}\overset{i}{\chi}_2(I_3-1)^2,$$

where

$$\overset{i}{W}\left(\overset{i}{I}_1,\overset{i}{I}_2\right) = W_i\left(\overset{i}{I}_1,\overset{i}{I}_2,1\right),\quad \overset{i}{\chi}_1\left(\overset{i}{I}_1,\overset{i}{I}_2\right) = \left.\frac{\partial W_i}{\partial I_3}\right|_{I_3=1},$$

$$\overset{i}{\chi}_2\left(\overset{i}{I}_1,\overset{i}{I}_2\right) = \left.\frac{\partial^2 W_i}{\partial I_3^2}\right|_{I_3=1},\quad i=1,2,3.$$

From these expressions with the help of the ordinary procedure we get the following equations:

$$\tfrac{1}{2}\sqrt{I_3}\mathbf{Q} = \mathbf{g}\overset{1}{C}_1 + \left(\overset{1}{I}_1\mathbf{g}-\mathbf{G}^x\right)\overset{1}{C}_2 + \overset{1}{\sigma}I_3\mathbf{G}^{x-1}, \tag{3a}$$

$$\tfrac{1}{2}\sqrt{I_3}\mathbf{Q} = \frac{1}{I_3^{1/3}}\left(\mathbf{g}-\frac{\overset{1}{I}_1}{3}\mathbf{G}^{x-1}\right)\overset{2}{C}_1 + \frac{1}{I_3^{2/3}}\left(\overset{1}{I}_1\mathbf{g}-\mathbf{G}^x-\frac{2}{3}\overset{1}{I}_2\mathbf{G}^{x-1}\right)\overset{2}{C}_2+\overset{2}{\sigma}I_3\mathbf{G}^{x-1}, \tag{3b}$$

$$\tfrac{1}{2}\sqrt{I_3}\mathbf{Q} = (\mathbf{g}-I_3\mathbf{G}^{x-1})\overset{3}{C}_1 + \left(\overset{1}{I}_1\mathbf{g}-\mathbf{G}^x-2I_3\mathbf{G}^{x-1}\right)\overset{3}{C}_2+\overset{3}{\sigma}I_3\mathbf{G}^{x-1}, \tag{3c}$$

where

$$\overset{i}{\alpha}\left(\overset{i}{\sigma}-\overset{i}{\chi}_1\right) = I_3-1,\quad \overset{i}{\alpha} = 1/\overset{i}{\chi}_2,$$

$$\overset{i}{C}_j\left(\overset{i}{I}_1,\overset{i}{I}_2,I_3\right) = \overset{i}{k}_j+\overset{i}{p}_j(I_3-1)+\tfrac{1}{2}\overset{i}{q}_j(I_3-1)^2,$$

$$\overset{i}{k}_j\left(\overset{i}{I}_1,\overset{i}{I}_2\right) = \partial\overset{i}{W}/\partial\overset{i}{I}_j,\quad \overset{i}{p}_j\left(\overset{i}{I}_1,\overset{i}{I}_2\right) = \partial\overset{i}{\chi}_1/\partial\overset{i}{I}_j,$$

$$\overset{i}{q}_j\left(\overset{i}{I}_1,\overset{i}{I}_2\right) = \partial\overset{i}{\chi}_2/\partial\overset{i}{I}_j,\quad i=1,2,3,\quad j=1,2, \tag{4}$$

$\mathbf{Q}$ is the energy stress tensor, $\mathbf{g}$ is the metric tensor of undeformed volume. In the undeformed state eqns (3) lead to the relations

$$\tfrac{1}{2}\mathbf{Q} = \left[\overset{1}{k}_1(3,3)+2\overset{1}{k}_2(3,3)+\overset{1}{\chi}_1(3,3)\right]\mathbf{g},$$

$$\tfrac{1}{2}\mathbf{Q} = \overset{2}{\chi}_1(3,3)\mathbf{g},\quad \tfrac{1}{2}\mathbf{Q} = \overset{3}{\chi}_1(3,3)\mathbf{g},$$

the results of which for natural (without stress) initial configuration are

$$\overset{1}{k}_1(3,3)+2\overset{1}{k}_2(3,3)+\overset{1}{\chi}_1(3,3) = 0,\quad \overset{2}{\chi}_1(3,3) = \overset{3}{\chi}_1(3,3) = 0. \tag{5}$$

Reduction of the expressions (3) to the linear elasticity, which can be easily carried out using eqns (2) and keeping only linear terms for deformations, gives the following equalities for generalized elasticity moduli, Lamé constant Λ and shear modulus G:

$$\Lambda = 4\left.\left[\frac{\partial}{\partial \overset{1}{I}_1}\left(\overset{1}{k}_1+2\overset{1}{k}_2+\overset{1}{\chi}_1\right)+2\frac{\partial}{\partial \overset{1}{I}_2}\left(\overset{1}{k}_1+2\overset{1}{k}_2+\overset{1}{\chi}_1\right)-\left(\overset{1}{k}_1+\overset{1}{k}_2\right)+\left(\overset{1}{p}_1+2\overset{1}{p}_2+\overset{1}{\chi}_2\right)\right]\right|_{\mathbf{G}^x=\mathbf{g}},$$

$$G = 2\left.\left(\overset{1}{k}_1+\overset{1}{k}_2\right)\right|_{\mathbf{G}^x=\mathbf{g}}; \tag{6a}$$

$$\Lambda = 4\left[\overset{2}{\chi}_2 - \frac{1}{3}\left(\overset{2}{k}_1+\overset{2}{k}_2\right)\right]\Bigg|_{G^x=g}, \quad G = 2\left(\overset{2}{k}_1+\overset{2}{k}_2\right)\Bigg|_{G^x=g}; \tag{6b}$$

$$\Lambda = 4\left[\overset{3}{\chi}_2 - \left(\overset{3}{k}_1+\overset{3}{k}_2\right)\right]\Bigg|_{G^x=g}, \quad G = 2\left(\overset{3}{k}_1+\overset{3}{k}_2\right)\Bigg|_{G^x=g}. \tag{6c}$$

From the expressions (6b) and (6c) it follows that

$$B = \Lambda + \tfrac{2}{3}G = 4\overset{2}{\chi}_2\Big|_{G^x=g},$$

$$\nu = \frac{\Lambda}{2(\Lambda+G)} = \frac{3\overset{2}{\chi}_2 - \left(\overset{2}{k}_1+\overset{2}{k}_2\right)}{6\overset{2}{\chi}_2 + \left(\overset{2}{k}_1+\overset{2}{k}_2\right)}\Bigg|_{G^x=g}, \quad \nu = \frac{\overset{3}{\chi}_2 - \left(\overset{3}{k}_1+\overset{3}{k}_2\right)}{2\overset{3}{\chi}_2 - \left(\overset{3}{k}_1+\overset{3}{k}_2\right)}\Bigg|_{G^x=g}, \tag{7}$$

where B and ν are the volumetric modulus and Poisson ratio, respectively. In turn it follows from these relations that if $\overset{2}{\chi}_2(\overset{3}{\chi}_2) \Rightarrow \infty$ then $B \Rightarrow \infty$ and $\nu \Rightarrow 0.5$. Note that the last equality has been obtained by Cescotto and Fonder (1975). As the above noted parameters $\overset{i}{k}_j, \overset{i}{p}_j, \overset{i}{q}_j, \overset{i}{\chi}_j$, $i = 1, 2, 3$, $j = 1, 2$ in eqns (3) can be obtained from the elastic potential expressions or from experiments [see Rogovoy (1988) and Kozhevnikova *et al.* (1983)]. It should be pointed out that, if the constitutive equation with parameters determined experimentally is energetically admissible, then the equalities following from the relations (4) must be fulfilled:

$$\frac{\partial \overset{i}{k}_1}{\partial \overset{i}{I}_2} = \frac{\partial \overset{i}{k}_2}{\partial \overset{i}{I}_1}, \quad \frac{\partial \overset{i}{p}_1}{\partial \overset{i}{I}_2} = \frac{\partial \overset{i}{p}_2}{\partial \overset{i}{I}_1}, \quad \frac{\partial \overset{i}{q}_1}{\partial \overset{i}{I}_2} = \frac{\partial \overset{i}{q}_2}{\partial \overset{i}{I}_1}.$$

Let us show the principal possibility for determination of some generalized elasticity moduli from the three simple experiments and their correspondence to one or another group of invariants in the sense of simplicity of elasticity moduli identification.

Simple shear

In this process the material point, defined before deformation by radius vector $\mathbf{r} = x^i\mathbf{e}_i$ ($\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, $i, j = 1, 2, 3$, δ_{ij} is the Kronecker delta), in the actual configuration occupies the place $\mathbf{R} = (x^1+sx^2)\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3$ (s is a shear parameter). Then the basis vectors, components of metric tensor and determinant of its covariant components are written in the initial and actual configurations as:

$$\mathbf{r}_i = \partial\mathbf{r}/\partial x^i: \quad \mathbf{r}_1 = \mathbf{e}_1, \quad \mathbf{r}_2 = \mathbf{e}_2, \quad \mathbf{r}_3 = \mathbf{e}_3;$$

$$g_{ii} = g^{ii} = 1, \quad g_{ij} = g^{ij} = 0, \quad i \neq j; \quad g = 1;$$

$$\mathbf{R}_i = \partial\mathbf{R}/\partial x^i: \quad \mathbf{R}_1 = \mathbf{e}_1, \quad \mathbf{R}_2 = s\mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{R}_3 = \mathbf{e}_3;$$

$$\left\{\begin{matrix} G_{11} = 1 & G_{12} = s & G_{13} = 0 \\ G_{21} = s & G_{22} = 1+s^2 & G_{23} = 0 \\ G_{31} = 0 & G_{32} = 0 & G_{33} = 1 \end{matrix}\right\}, \quad \left\{\begin{matrix} G^{11} = 1+s^2 & G^{12} = -s & G^{13} = 0 \\ G^{21} = -s & G^{22} = 1 & G^{23} = 0 \\ G^{31} = 0 & G^{32} = 0 & G^{33} = 1 \end{matrix}\right\},$$

$G = 1$. From these relations $\overset{1}{I}_1 = g^{ij}G_{ij} = 3+s^2$, $\overset{1}{I}_2 = (G/g)g_{ij}G^{ij} = 3+s^2$, $I_3 = 1$ and $\overset{1}{I}_j = \overset{2}{I}_j = \overset{3}{I}_j$, $j = 1, 2$. So taking into account the relations (3) we have the following nonzero true stress tensor $\mathbf{T} = t^{ij}\mathbf{e}_i\mathbf{e}_j$ components:

$$t^{11} = 2\left(\overset{1}{k}_1+2\overset{1}{k}_2+\overset{1}{\chi}_1\right)+2\left(\overset{1}{k}_1+\overset{1}{k}_2\right)s^2, \quad t^{12} = 2\left(\overset{1}{k}_1+\overset{1}{k}_2\right)s,$$

$$t^{22} = 2\left(\overset{1}{k}_1+2\overset{1}{k}_2+\overset{1}{\chi}_1\right), \quad t^{33} = 2\left(\overset{1}{k}_1+2\overset{1}{k}_2+\overset{1}{\chi}_1\right)+2\overset{1}{k}_2s^2; \tag{8a}$$

$$t^{11} = 2\overset{2}{\chi}_1+\tfrac{2}{3}\left(2\overset{2}{k}_1+\overset{2}{k}_2\right)s^2, \quad t^{12} = 2\left(\overset{2}{k}_1+\overset{2}{k}_2\right)s,$$

$$t^{22} = 2\overset{2}{\chi}_1-\tfrac{2}{3}\left(\overset{2}{k}_1+2\overset{2}{k}_2\right)s^2, \quad t^{33} = 2\overset{2}{\chi}_1-\tfrac{2}{3}\left(\overset{2}{k}_1-\overset{2}{k}_2\right)s^2; \tag{8b}$$

$$t^{11} = 2\overset{3}{\chi}_1+2\left(\overset{3}{k}_1+\overset{3}{k}_2\right)s^2, \quad t^{12} = 2\left(\overset{3}{k}_1+\overset{3}{k}_2\right)s,$$

$$t^{22} = 2\overset{3}{\chi}_1, \quad t^{33} = 2\overset{3}{\chi}_1+2\overset{3}{k}_2s^2. \tag{8c}$$

In this problem the stresses t^{11}, t^{22} and t^{33} can be measured experimentally so the quantities $\overset{i}{k}_1$, $\overset{i}{k}_2$ and $\overset{i}{\chi}_1$ can be defined. The simplest relations follow from expressions (8c) for $\overset{3}{k}_1$, $\overset{3}{k}_2$ and $\overset{3}{\chi}_1$.

Hydrostatic stress state

In this process

$$\mathbf{r} = x^i\mathbf{e}_i, \quad \mathbf{R} = \lambda x^i\mathbf{e}_i$$

and then

$$\mathbf{r}_i = \mathbf{e}_i, \quad \mathbf{r}_2 = \mathbf{e}_2, \quad \mathbf{r}_3 = \mathbf{e}_3;$$

$$g_{ii} = g^{ii} = 1, \quad g_{ij} = g^{ij} = 0, \quad i \neq j; \quad g = 1;$$

$$\mathbf{R}_1 = \lambda\mathbf{e}_1, \quad \mathbf{R}_2 = \lambda\mathbf{e}_2, \quad \mathbf{R}_3 = \lambda\mathbf{e}_3;$$

$$G_{ii} = \lambda^2, \quad G^{ii} = \frac{1}{\lambda^2}, \quad G_{ij} = G^{ij} = 0, \quad i \neq j; \quad G = \lambda^6;$$

$$\overset{1}{I}_1 = 3\lambda^2, \quad \overset{1}{I}_2 = 3\lambda^4, \quad \overset{1}{I}_3 = \lambda^6; \quad \overset{2}{I}_1 = \overset{2}{I}_2 = 3$$

and from the expressions (5)

$$\overset{2}{\chi}_1 = 0; \quad \overset{3}{I}_1 = 3\lambda^2-(\lambda^6-1), \quad \overset{3}{I}_2 = 3\lambda^4-2(\lambda^6-1);$$

$$t^{11} = t^{22} = t^{33} = 2\frac{\overset{1}{C}_1}{\lambda}+4\overset{1}{C}_2\lambda+2\lambda^3\left[\overset{1}{\chi}_1+\overset{1}{\chi}_2(\lambda^6-1)\right]; \tag{9a}$$

$$t^{11} = t^{22} = t^{33} = 2\lambda^3\overset{2}{\chi}_2(3,3)(\lambda^6-1); \tag{9b}$$

$$t^{11} = t^{22} = t^{33} = 2\overset{3}{C}_1\left(\frac{1}{\lambda}-\lambda^3\right)+4\overset{3}{C}_2\lambda\left(1-\lambda^2\right)+2\lambda^3\left[\overset{3}{\chi}_1+\overset{3}{\chi}_2(\lambda^6-1)\right]. \tag{9c}$$

Here, the quantity $\overset{2}{\chi}_2(3,3)$ is easily defined from the relation (9b).

Uni-axial strain state

The process of the cylindrical body upsetting in a closed volume corresponds to this problem. The kinematic and force characteristics of the process for cylindrical frame of reference $q^1 = r$, $q^2 = \varphi$, $q^3 = z$, where q^i are generalized coordinates, are as follows:

$$\mathbf{r} = r\mathbf{r}_1 + z\mathbf{r}_3, \quad \mathbf{R} = r\mathbf{r}_1 + \lambda z\mathbf{r}_3;$$

$$\mathbf{r}_1 = \mathbf{e}_1, \quad \mathbf{r}_2 = r\mathbf{e}_2, \quad \mathbf{r}_3 = \mathbf{e}_3, \quad |\mathbf{e}_i| = 1, \quad i = 1,2,3;$$

$$g_{11} = g^{11} = g_{33} = g^{33} = 1, \quad g_{22} = r^2, \quad g^{22} = \frac{1}{r^2}, \quad g_{ij} = g^{ij} = 0, \quad i \neq j; \quad g = r^2;$$

$$\mathbf{R}_1 = \mathbf{e}_1, \quad \mathbf{R}_2 = r\mathbf{e}_2, \quad \mathbf{R}_3 = \lambda\mathbf{e}_3, \quad G_{11} = G^{11} = 1, \quad G_{22} = r^2, \quad G_{33} = \lambda^2,$$

$$G^{22} = 1/r^2, \quad G^{33} = 1/\lambda^2, \quad G_{ij} = G^{ij} = 0, \quad i \neq j; \quad G = (r\lambda)^2;$$

$$\overset{1}{I}_1 = 2+\lambda^2, \quad \overset{1}{I}_2 = 1+2\lambda^2, \quad I_3 = \lambda^2;$$

$$\overset{2}{I}_1 = \frac{2+\lambda^2}{\lambda^{2/3}}, \quad \overset{2}{I}_2 = \frac{1+2\lambda^2}{\lambda^{4/3}}, \quad \overset{3}{I}_1 = \overset{3}{I}_2 = 3$$

and from the relations (5) $\overset{3}{\chi}_1 = 0$;

$$t^{11} = t^{22} = 2\frac{\overset{1}{C}_1}{\lambda} + 2\left(\frac{1}{\lambda} + \lambda\right)\overset{1}{C}_2 + 2\lambda\left[\overset{1}{\chi}_1 + \overset{1}{\chi}_2(\lambda^2 - 1)\right],$$

$$t^{33} = 2\lambda\left(\overset{1}{C}_1 + 2\overset{1}{C}_2\right) + 2\lambda\left[\overset{1}{\chi}_1 + \overset{1}{\chi}_2(\lambda^2 - 1)\right]; \tag{10a}$$

$$t^{11} = t^{22} = \frac{2}{3\lambda^{5/3}}\left(\overset{2}{C}_1 + \frac{\overset{2}{C}_2}{\lambda^{2/3}}\right)(1-\lambda^2) + 2\lambda\left[\overset{2}{\chi}_1 + \overset{2}{\chi}_2(\lambda^2 - 1)\right],$$

$$t^{33} = -\frac{4}{3\lambda^{5/3}}\left(\overset{2}{C}_1 + \frac{\overset{2}{C}_2}{\lambda^{2/3}}\right)(1-\lambda^2) + 2\lambda\left[\overset{2}{\chi}_1 + \overset{2}{\chi}_2(\lambda^2 - 1)\right]; \tag{10b}$$

$$t^{11} = t^{22} = 2\frac{1-\lambda^2}{\lambda}\left[\overset{3}{C}_1(3,3,\lambda^2) + \overset{3}{C}_2(3,3,\lambda^2)\right] + 2\lambda\overset{3}{\chi}_2(3,3)(\lambda^2 - 1),$$

$$t^{33} = 2\lambda\overset{3}{\chi}_2(3,3)(\lambda^2 - 1). \tag{10c}$$

In this problem, stresses t^{11} and t^{33} are measured and the quantity $\overset{3}{\chi}_2(3,3)$ and the complex $\overset{3}{C}_1(3,3,\lambda^2) + \overset{3}{C}_2(3,3,\lambda^2)$ are easily defined from the expression (10c).

Thus, assuming the simplicity of the generalized elasticity moduli identification (or part of them) on the basis of experimental data as a criterion for selection of one or other group of invariants, used in the elastic potential W, we given the preference to invariants (1c) for the above examined problems. And in the following section of this paper we use the state equations (3c), (4) without index 3 over the letters.

Of course, from the relations (8)–(10), taking into account expressions (5), we get zero stress for the undeformed state. In the case of incompressible material we have $\lambda = 1$ for eqns (9) and (10). And again taking into account the expressions (5) we find that $\overset{i}{\chi}_2 = \infty$, $i = 1, 2, 3$. [For $i = 2$ this result was reported by Kozhevnikova *et al.* (1983) and by Rogovoy (1988).] Then from the relations (4) we conclude that $\overset{i}{\alpha} = 0$ and it leads to $I_3 - 1 = 0$. So the variable $\overset{i}{\sigma}$ in the expressions (3) becomes a new unknown quantity defined from the solution of the problem. It follows from the above consideration that the constitutive equations (3), (4) are justified both for the slightly compressible and incompressible materials.

3. VARIATIONAL FORMULATION OF THE PROBLEM AND NUMERICAL IMPLEMENTATION

For solving non-linear elastic problems a functional for incompressible and nearly incompressible materials is used (Rogovoy, 1988):

$$He(H,\mathbf{u}) = \int_{V_0} \left\{ AH(I_3-1) - A^2\frac{\alpha}{2}(H-\chi^*)^2 + W(I_1,I_2) + \tfrac{1}{2}(k_1+k_2)[(I_3-1)-A\alpha(H-\chi^*)]^2 - \rho^0\mathbf{K}\cdot\mathbf{u} \right\} \mathrm{d}V_0 - \int_{S_p^0} \mathbf{P}\cdot\mathbf{u}\,\mathrm{d}S_p^0. \quad (11)$$

In the linear case it induces the Herrmann (1965) functional. In expression (11) $\mathbf{u}$ is the displacement vector,

$$AH = \sigma, \quad A\chi^* = \chi_1, \quad A = \frac{2(k_1+k_2)}{2-\alpha(k_1+k_2)},$$

V_0 and S_p^0 are respectively the unstrained volume with volume forces $\mathbf{K}$ and density ρ^0 and the surface where forces $\mathbf{P}$ are applied. The functional (11) varies in H and $\mathbf{u}$. Rogovoy (1988) has shown that the variational equation is equivalent to a continuum mechanics equation system and has performed an analytical check on the constitutive equations (3c), (4) and the functional (11).

The finite element method has been used for the calculations. The generalized elasticity moduli in (3c), (4) are assumed to be constant, so $\chi_1 = 0$ and $\chi^* = 0$. Varying the functional (11) in $\mathbf{u}$ and approximating the displacement vector components by the shape function ψ:

$$u_p = \psi_n u_p^n, \quad u^k = \psi_m u^{mk},$$

where n, m are the node element numbers, u_p^n, u^{mk} are the covariant and contravariant displacement vector components in n and m nodes, we get for element $\overset{(e)}{V_0}$:

$$\left\{ \int_{\overset{(e)}{V_0}} [2\sigma^{ij}(A_{ijn}^p + B_{ijkmn}^p u^{mk}) - \rho^0 K^p \psi_n]\,\mathrm{d}\overset{(e)}{V_0} - \int_{\overset{(e)}{S_p^0}} P^p \psi_n\,\mathrm{d}\overset{(e)}{S_p^0} \right\} \delta u_p^n = 0, \quad (12)$$

where

$$\sigma^{ij} = \tfrac{1}{2}\sqrt{I_3}Q^{ij} + (k_1+k_2)[(I_3-1)-A\alpha H]I_3 G^{ij},$$
$$A_{ijn}^p = g_i^p \psi_{n,j} - \psi_n \Gamma_{ij}^p, \quad \psi_{n,j} = \partial\psi_n/\partial q^j,$$
$$B_{ijkmn}^p = g_k^p \psi_{m,i}\psi_{n,j} - \psi_{m,i}\psi_n\Gamma_{jk}^p + \psi_m\psi_{n,j}\Gamma_{ki}^p - \psi_m\psi_n\Gamma_{ki}^s\Gamma_{js}^p, \quad (13)$$

Γ_{ij}^k are the 2nd order Kristoffel symbols. Varying the functional (11) in H and approximating H as $H = \varphi_n H^n$, we restrict ourselves to

$$\left\{ \int_{\overset{(e)}{V_0}} M[(I_3-1) - A\alpha\varphi_m H^m]\varphi_n\,\mathrm{d}\overset{(e)}{V_0} \right\} \delta H^n = 0, \quad (14)$$

where

$$M = A[1-\alpha(k_1+k_2)].$$

The system of equations for the functional (11) is non-linear, since σ^{ij} is explicitly dependent both on $\mathbf{u}$ and H, and P^p and I_3 only on $\mathbf{u}$. The dependence of P^p on $\mathbf{u}$ is associated with reducing the force F^{p}, specified in the current configuration, on the undeformed surface S_{p}^{0}:

$$P^{\mathrm{p}} = \sqrt{I_3 G^{ij} n_i n_j} F^{\mathrm{p}}, \tag{15}$$

where n_i are covariant components of the unit vector $\mathbf{n}$, normal to the undeformed surface S_{p}^{0}. Let us linearize eqns (12)–(15). Let

$$\begin{aligned} u^{mk} &= \overset{0}{u}{}^{mk} + \xi^{mk}, \quad H^m = \overset{0}{H}{}^m + \eta^m, \\ K^{\mathrm{p}} &= \overset{0}{K}{}^{\mathrm{p}} + B^{\mathrm{p}}, \quad F^{\mathrm{p}} = \overset{0}{F}{}^{\mathrm{p}} + S^{\mathrm{p}}. \end{aligned} \tag{16}$$

Since the values ξ^{mk} and η^m are small, their squared values can be neglected. As was mentioned above, σ^{ij} can be represented as

$$\sigma^{ij} = \sigma^{ij}(u^{mk}, H^m).$$

Substituting here the expressions (16) and following Oden (1972) we make a series expansion in terms of ξ^{mk} and η^m, having preserved only the linear terms:

$$\sigma^{ij}(u^{mk}, H^m) = \overset{0}{\sigma}{}^{ij} + \overset{0}{\sigma}{}^{ij}_{,\xi^{mk}} \xi^{mk} + \overset{0}{\sigma}{}^{ij}_{,\eta^m} \eta^m. \tag{17}$$

Taking into account that $P^{\mathrm{p}} = P^{\mathrm{p}}(u^{mk})$, and transforming the expression (15) in the same manner, we get

$$P^{\mathrm{p}} = \overset{0}{P}{}^{\mathrm{p}} + \overset{0}{P}{}^{\mathrm{p}}_{\mathrm{s}} + \overset{0}{P}{}^{\mathrm{p}}_{,\xi^{mk}} \xi^{mk}, \tag{18}$$

where

$$\overset{0}{P}{}^{\mathrm{p}} = \sqrt{\overset{0}{I}_3 \overset{0}{G}{}^{ij} n_i n_j} \overset{0}{F}{}^{\mathrm{p}}, \quad \overset{0}{P}{}^{\mathrm{p}}_{\mathrm{s}} = \sqrt{\overset{0}{I}_3 \overset{0}{G}{}^{ij} n_i n_j} S^{\mathrm{p}},$$

$$\overset{0}{P}{}^{\mathrm{p}}_{,\xi^{mk}} = \frac{\left(\overset{0}{I}_{3,\xi^{mk}} \overset{0}{G}{}^{ij} + \overset{0}{I}_3 \overset{0}{G}{}^{ij}_{,\xi^{mk}}\right) n_i n_j}{2\sqrt{\overset{0}{I}_3 \overset{0}{G}{}^{ij} n_i n_j}} \left(\overset{0}{F}{}^{\mathrm{p}} + S^{\mathrm{p}}\right). \tag{19}$$

Substituting the expressions (17)–(19) into (12) and (14), taking into account (16), omitting terms equal to zero and preserving only the linear terms ξ^{mk} and η^m, we have:

$$\begin{aligned} &\left\{ \int_{\overset{(e)}{V_0}} [2\overset{0}{\sigma}{}^{ij} B^{\mathrm{p}}_{ijkmn} + 2\overset{0}{\sigma}{}^{ij}_{,\xi^{mk}} (A^{\mathrm{p}}_{ijn} + B^{\mathrm{p}}_{ijsln} \overset{0}{u}{}^{ls})] \, \mathrm{d}\overset{(e)}{V}_0 \right. \\ &\left. - \int_{\overset{(e)}{S^0_{\mathrm{p}}}} \overset{0}{P}{}^{\mathrm{p}}_{,\xi^{mk}} \psi_n \, \mathrm{d}\overset{(e)}{S}{}^0_{\mathrm{p}} \right\} \xi^{mk} + \left\{ \int_{\overset{(e)}{V_0}} 2\overset{0}{\sigma}{}^{ij}_{,\eta^m} (A^{\mathrm{p}}_{ijn} + B^{\mathrm{p}}_{ijsln} \overset{0}{u}{}^{ls}) \, \mathrm{d}\overset{(e)}{V}_0 \right\} \eta^m \\ &- \int_{\overset{(e)}{V_0}} \rho^0 B^{\mathrm{p}} \psi_n \, \mathrm{d}\overset{(e)}{V}_0 - \int_{\overset{(e)}{S^0_{\mathrm{p}}}} \overset{0}{P}{}^{\mathrm{p}}_{s} \psi_n \, \mathrm{d}\overset{(e)}{S}{}^0_{\mathrm{p}} = 0, \quad (\delta u^n_{\mathrm{p}}), \\ &\int_{\overset{(e)}{V_0}} M \left(\overset{0}{I}_{3,\xi^{mk}} \xi^{mk} - A\alpha \varphi_m \eta^m \right) \varphi_n \, \mathrm{d}\overset{(e)}{V}_0 = 0, \quad (\delta H^n). \end{aligned}$$

Summing the expressions for all elements, we get a complete resolution system for the functional (11). The specification of the equation system for axisymmetrical problems was performed by Kozhevnikova *et al.* (1983).

The stiffness matrix is symmetric in the absence of surface tractions; that helps with solving problems. A program was developed and tested, using triangular cylindrical finite elements with a square approximation to the displacement field, and linear functions for the mean pressure and incremental load procedure. In contact zones the conditions of non-penetration and non-positiveness of normal pressure have been introduced.

4. NUMERICAL RESULTS AND DISCUSSION

The whole body of mathematical expressions developed above has been applied to void evolution analysis around rigid spherical inclusion in an incompressible elastomeric matrix. The matrix properties are supposed to be neo-Hookean characterized by the elastic potential ($k_1 = 0.05$ MPa, $k_2 = 0$), in accordance with real material behavior in the deformation region of about 100–200%.

The inclusion is compacted in the spherical cavity of the elastomer (Fig. 1). No bonding between matrix and inclusion and no friction at the contact surface are assumed. Cylinder ends are stretched, vertical displacements W_0 being constant and radial ones not constrained. The stress–strain state of the matrix is supposed to be sensitive to lateral pressure after the detachment has occurred. So the stress–strain analysis was performed under various lateral pressures. The measure of external deformation is chosen as $\varepsilon_z^c = (\lambda_c^2 - 1)/2$, where $\lambda_c = (L+2W_0)/L$ is the elongation factor for the point C (see Fig. 1). As the problem is symmetrical, one quarter of the body has been analysed and Fig. 2 illustrates the boundary conditions and the model of the finite element mesh used in the calculations. The latter is obtained by transformation of the standard finite element mesh to the computational region with condensation of elements near the inclusion and symmetry axis of the body as shown in Fig. 2. In the present calculation it is assumed that $N = 7$, $M = 27$ for the standard finite element mesh, steps for W_0 and p being equal to 0.05 mm and 0.005 MPa, respectively.

Figure 3 represents profiles of vacuoles under external deformations $\varepsilon_z^c = 0.25$, 0.4 and 0.6 with lateral pressures being equal to (a) zero and (b) $p = 0.1$ MPa. On the void surface we have marked one and the same material point to show the character of displacement.

The plots in Fig. 4 show the principal deformation (curve 1) and stress (curve 2) distributions along the vacuole contour and principal stresses (curve 3) normal to the vacuole contour, reduced to the unstrained volume coordinates, depicted for external deformation $\varepsilon_z^c = 0.6$, with lateral pressures (a) 0 and (b) 0.1 MPa.

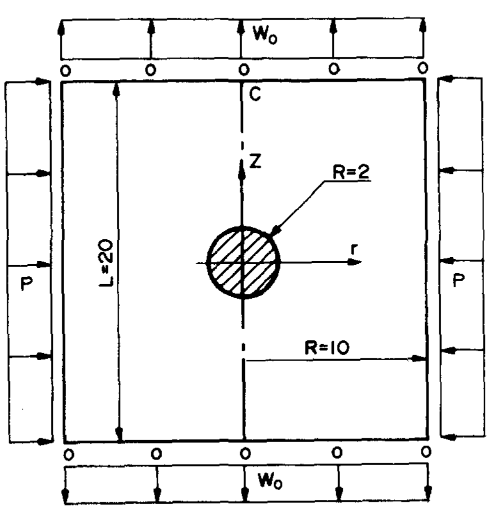

Fig. 1. Modelling problem (in mm).

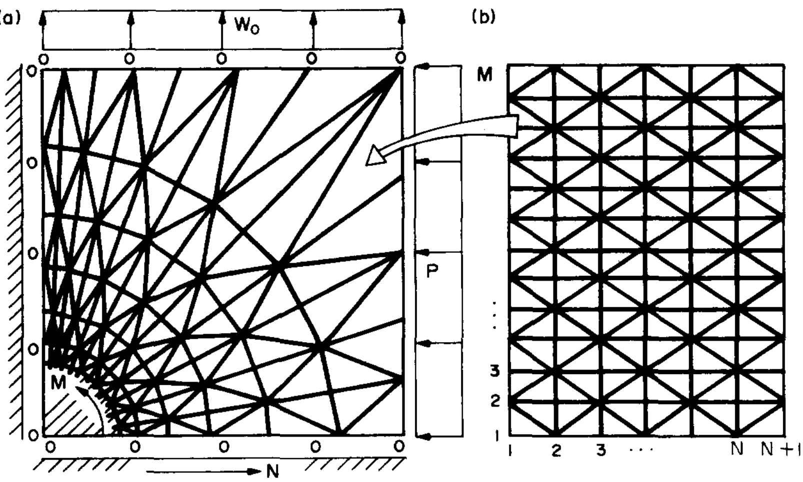

Fig. 2. Boundary conditions and model of the finite element mesh used in calculation (a) and obtained by transformation of the standard finite element mesh (b).

From curve 1, one can clearly see that in the absence of lateral pressure the maximum local strain along the vacuole profile appears in the equator zone and reaches 160% (right-hand scale). Under lateral pressure the maximum local strain shifts to the line separating the contact and free vacuole surfaces, the maximum value of strain being decreased a factor of 1.3. Lateral pressure influences stress (curves 2, 3) rather than deformation. The maximum tensile stresses for the contour (point A) are located near the line separating the contact and free vacuole surfaces both in the presence and absence of lateral pressure. In the latter case this stress is twice as high (left-hand scale). In the vacuole pole compressible stresses take place in both cases. In the presence of lateral pressure they are more intense. Compressible contact stresses (curve 3) are more than twice as high under external pressure, the contact zone being larger in this case.

The performed analysis of matrix detachment from inclusion has shown that the zone, where the matrix is separated from the sphere surface, seems to be the most liable for cracking. The experimental data by Gent and Park (1984) corroborate this conclusion. It has been established that superposition of external pressure decreases maximal extensional deformation, as well as maximal tensile stress, at the void walls under the same external

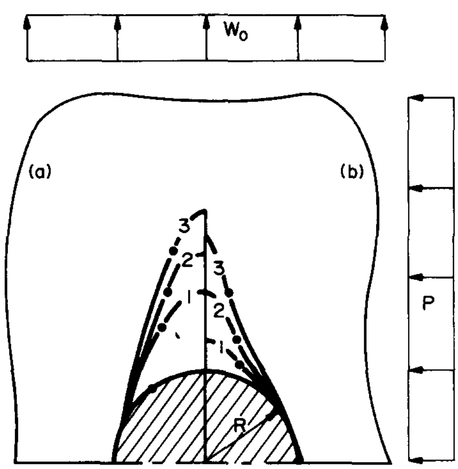

Fig. 3. Profiles of vacuoles under external deformations $\varepsilon_z^c = 0.25$ (curve 1), 0.4 (curve 2) and 0.6 (curve 3) with lateral pressures being equal to (a) zero and (b) $p = 0.1$ MPa. The scale of quantity $|\mathbf{u}|/R$ is 2 : 1.

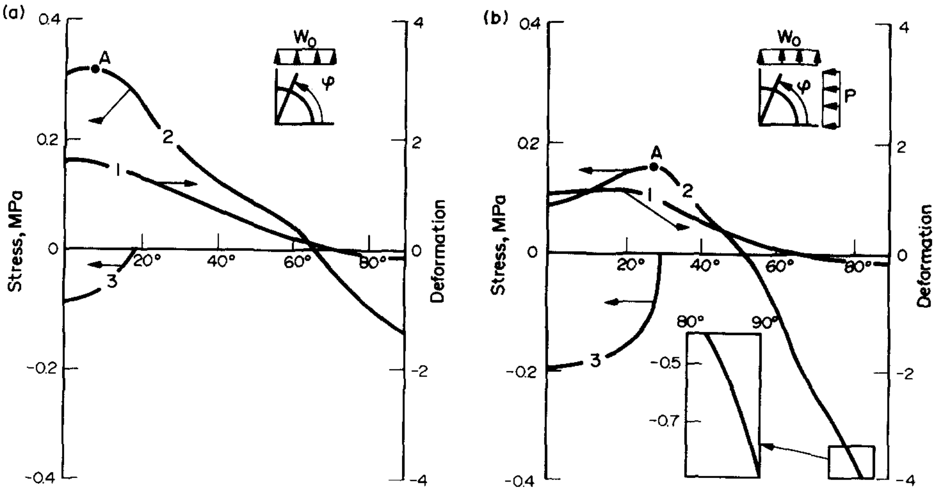

Fig. 4. The principal deformation (curve 1) and stress (curve 2) distributions along the vacuole contour and principal stresses (curve 3) normal to the vacuole contour, reduced to the unstrained volume coordinates, for external deformation $\varepsilon_z^c = 0.6$ with lateral pressures being equal to (a) zero and (b) $p = 0.1$. MPa.

deformation. It is one of the factors promoting the overall increase in both strength and deformation of the composite stretched under pressure.

5. CONCLUSION

The method for solving boundary problems of finite elasticity is developed. The stress–strain state of an elastomer detached from spherical inclusion is examined. The analysis partly clarifies the strengthening effect of external pressure on the elastomeric granular composites.

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